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Gravitational Energy and Momentum: A Tensorial Approach

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A tensorial expression for localized gravitational energy-momentum is delineated as an integral part of the energy-momentum tensor. A bona fide conservation law of the total energy-momentum tensor is obtained in the geodesic-nonrotating coordinates, in which the covariant divergencelessness of the energy-momentum tensor reads, globally, as ordinary divergencelessness. The integral gravitational energy in the exterior of a spherically symmetric source is calculated based on this tensorial relativistic expression. For an ordinary star, such as the sun, it coincides with the Newtonian value up to six digits.

1. INTRODUCTION

In recent papers (Nissani and Leibowitz, 1988, 1989, 1990; Carmeli *et al.*, 1990) a special class of geodesic coordinate systems has been delineated and investigated. In these preferred frames the laws of physics assume locally their special relativistic form, whereas the energy-momentum tensor satisfies a global conservation law, i.e., its ordinary divergence vanishes. This global conservation law does away with the need for a kind of energy not included in the energy-momentum tensor, and at the same time bears on the possibility of defining gravitational energy in a tensorial form, as an integral part of the energy-momentum tensor.

The gravitational energy-momentum problem has been tackled by many authors (e.g., Einstein, 1916; Rosen, 1940; Landau and Lifshitz, 1951; Bergmann and Thomson, 1953; Moller, 1958; Goldberg, 1958; Komar, 1959; Cornish, 1964; Trautman, 1967; Penrose, 1982; Kovacs, 1985), and a variety of pseudo-tensors has been proposed as the expression of the gravitational energy-momentum. It is, however, generally agreed that no completely

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satisfactory solution to this problem has been offered (see, for example, Maddox, 1985).

The present paper introduces a true tensorial approach to the age-old problem of gravitational energy in the framework of general relativity. A tensorial expression for localized gravitational energy-momentum is obtained. Based on this tensorial expression, the space integral of the gravitational energy in the surroundings of a star turns out to have a value which is, for ordinary stellar objects, in high agreement with the Newtonian value.

The need and rationalization for such a tensorial expression relies on the existence and properties of the special frames mentioned above. We shall therefore recap those results of our previous study of the preferred geodesic frames which pertain to the main thrust of the present work.

Section 2 discusses briefly the geodesic-nonrotating frames along with their implication for the problem of the existence and nature of the gravitational energy. The basic building blocks of our tensorial approach, the fundamental tetrad vectors, are studied in Section 3. A decomposition of the Einstein tensor, induced by the fundamental tetrad, is exhibited, and the field equations are spelled out in Section 4. The behavior of the field equations under Lorentz transformations of the tetrad is studied in Section 5. An exact solution of the field equations is derived in Section 6 for the case of a spherically symmetric source, and the resulting invariant value of the gravitational energy is compared with the Newtonian value. Section 7 is devoted to concluding remarks.

2. NONROTATING COORDINATES

Any given symmetric tensor singles out, by a prescription outlined in Nissani and Leibowitz (1990), a class of coordinate systems, entitled "adapted to the given tensor," in which the ordinary divergence of the tensor vanishes. Specifically, if $C^{\alpha\beta}$ is a symmetric tensor and $\xi^{(\mu)}$ are four scalar functions, then the following ten integral functionals may be defined:

$$I^{(\mu)(\nu)} = \int_V \sqrt{-g}\, C^{\alpha\beta} \xi^{(\mu)}_{,\alpha} \xi^{(\nu)}_{,\beta}\, d^4x \tag{1}$$

where V is an arbitrary region of spacetime. We now consider the action principle based on these integrals, namely, we require a stationary value of these functionals with respect to variations of the scalar functions $\xi^{(\mu)}$, subject to the constraint that the variations vanish on the boundary of the integration region. The ensuing Lagrange equations

$$(C^{\alpha\beta}\xi^{(\nu)}_{,\beta})_{;\alpha} = 0 \tag{2}$$

suggest passing to the coordinates defined by the four scalars $\xi^{(\mu)}$,

$$x'^{\mu} = \xi^{(\mu)}(x) \tag{3}$$

In the new coordinates, equation (2) assumes the form

$$(\sqrt{-g'}\, C'^{\alpha\beta})_{,\beta} = 0 \tag{4}$$

viz., a global continuity equation.

Thus, an arbitrary symmetric tensor satisfies a global continuity equation in those coordinate systems where the integrals of its components over any arbitrary four-dimensional region of spacetime attain a stationary value.

Particular examples of adapted frames are the harmonic coordinates, where the symmetric tensor is the metric tensor and equation (4) is simply the deDonder condition, and the nonrotating coordinates (Nissani and Leibowitz, 1988, 1989, 1990; Carmeli *et al.*, 1990), where the symmetric tensor is the energy-momentum tensor.

In order to assign some tangible physical significance to this "conservation law," one has to pose the question of the conditions on the symmetric tensor under which there exist adapted frames which are geodesic with respect to an arbitrary observer. As was shown (Nissani and Lebowitz, 1989, 1990), a necessary and sufficient condition for a symmetric tensor $C^{\alpha\beta}$ to admit geodesic adapted coordinate systems with respect to any given observer is that its covariant divergence vanishes:

$$C^{\alpha\beta}{}_{;\beta} = 0 \tag{5}$$

This condition holds in the case of the energy-momentum tensor $T^{\alpha\beta}$ that appears in the Einstein field equations. The geodesic coordinates adapted to it, the geodesic-nonrotating coordinates, are therefore distinguished by the fact that in these frames:

(a) The laws of physics assume locally their special relativistic form.
(b) The energy-momentum tensor satisfies a global continuity equation.

We are therefore in the curved spacetime of general relativity in a condition analogous to that existing in the flat space of special relativity, where the law of energy-momentum conservation is valid in a special class of coordinate systems, the inertial frames. As was shown (Nissani and Leibowitz, 1988, 1989), this global conservation law leads, with the aid of the Gauss theorem, to an integral conservation law of satisfactory physical significance.

3. THE FUNDAMENTAL TETRAD

Having established the bona fide conservation of the energy-momentum tensor appearing in the Einstein field equations, there is no more obstacle to accepting that all forms of energy-momentum, whether they are attributed to matter or are ascribed to gravitational contribution, are incorporated into the conserved total energy-momentum tensor, which may be now written as a sum of matter and gravitational energy

$$T_T^{\alpha\beta} = T_M^{\alpha\beta} + T_G^{\alpha\beta} \tag{6}$$

Since the gravitational energy-momentum tensor T_G is assumed to be, along with the Einstein tensor G, a function of geometric elements, the splitting (6) induces a similar splitting of the Einstein tensor

$$G^{\alpha\beta} = G_M^{\alpha\beta} + G_G^{\alpha\beta} \tag{7}$$

with the Einstein field equations cast in the form

$$G_M^{\alpha\beta} = \kappa T_M^{\alpha\beta}, \qquad G_G^{\alpha\beta} = \kappa T_G^{\alpha\beta} \tag{8}$$

and, clearly,

$$G^{\alpha\beta} = \kappa T_T^{\alpha\beta} \tag{9}$$

Motivated by these considerations, we search for a mechanism to express the Einstein tensor as a sum of two tensors constructed of geometric elements. If both tensors are required to be at most of quasilinear second differential order in the underlying geometric structural elements, it seems impossible to take the metric tensor components alone as the basic elements. It is necessary therefore to introduce some fundamental quantities out of which the metric tensor is constructible.

As an ansatz, we consider a set of four mutually orthogonal and normalized vector fields φ_a^α ($a = 0, 1, 2, 3$), which will be called the fundamental tetrad. Here orthonormality is defined in terms of the inner product induced by the metric tensor:

$$\phi_a^\alpha \eta^{ab} \phi_b^\beta = g^{\alpha\beta} \tag{10}$$

where $\eta^{ab} = \mathrm{diag}[1, -1, -1, -1]$ is the Minkowski matrix. Alternatively, one may start with a set of linearly independent vector fields ϕ_a and interpret (10) as the definition of the metric tensor. In the sequel, tetrad indices will be raised and lowered with the aid of the Minkowski matrices η^{ab} and η_{ab}.

A simple heuristic counting argument lends some support to the consideration of the 16 components of the tetrad as the fundamental geometric objects. If we construe the 20 components of the gravitational and matter energy-momentum tensors as the physical data that determine the geometry of spacetime, then any pair of the three equations (9) and (10) constitutes, in view of the Bianchi identities, a system of 16 equations for the 16 components of the tetrad.

Having introduced the fundamental tetrad, the Ricci coefficients of rotation are defined as the tetrad components of the covariant derivatives of the tetrad vector fields

$$P_{abc} = \phi^{\alpha}_{b}\phi^{\beta}_{c}\phi_{a\alpha;\beta} \tag{11}$$

which, in view of (10), satisfy the following antisymmetry relation:

$$P_{abc} + P_{bac} = 0 \tag{12}$$

This relation reduces the independent scalars to 24. They are further subject to integrability conditions which will be written down in the next section.

4. THE FIELD EQUATIONS: DECOMPOSITION OF THE EINSTEIN TENSOR

The Ricci coefficients of rotation provide a useful representation of the Riemann curvature tensor. By (11)

$$P_{\gamma\alpha\beta}\phi^{\gamma}_{a} = \phi_{a\alpha;\beta} \tag{13}$$

where

$$P_{\gamma\alpha\beta} = \phi^{a}_{\gamma}\phi^{b}_{\alpha}\phi^{c}_{\beta}P_{abc} \tag{14}$$

is the tensorial version of the Ricci coefficients of rotation. Evaluating the commutator of the covariant derivatives of the fundamental tetrad yields [with the aid of (13)]:

$$R^{\alpha}{}_{\beta\gamma\delta} = (P^{\alpha}{}_{\beta\gamma;\delta} - P^{\alpha}{}_{\beta\delta;\gamma}) + (P^{\alpha}{}_{\rho\delta}P^{\rho}{}_{\beta\gamma} - P^{\alpha}{}_{\rho\gamma}P^{\rho}{}_{\beta\delta}) \tag{15}$$

The Ricci tensor therefore is given by

$$R_{\alpha\beta} = (P^{\gamma}{}_{\alpha\gamma;\beta} - P^{\gamma}{}_{\alpha\beta;\gamma}) + (P^{\gamma}{}_{\delta\beta}P^{\delta}{}_{\alpha\gamma} - P^{\gamma}{}_{\delta\gamma}P^{\delta}{}_{\alpha\beta}) \tag{16}$$

This decomposition of the Ricci tensor bears some resemblance to its expression in terms of the affine connections, in the sense that in both cases one gets a sum of a rotor part and a commutator part. The decomposition (16), however, is more suitable for our purposes, since it exhibits the Ricci tensor as a sum of two tensorial quantities.

The conventional approach to the problem of gravitational energy in general relativity (e.g., Landau and Lifshitz, 1951) associates the gravitational contribution to the energy-momentum with the commutator of the affine connections in the expression of the Ricci tensor. The remaining part—the rotor of the affine connections—is interpreted as the physical value of the material energy and momentum (excluding gravitational), as measured in geodesic frames in which the commutator part vanishes. This analogy gives a clue as to how to associate the decomposition (16) to energy and momentum. For reasons which will become clear later, the roles of the rotor and commutator parts in our approach are reversed, and we take

$$R_{G\alpha\beta}=(P^{\gamma}{}_{\alpha\gamma;\beta}-P^{\gamma}{}_{\alpha\beta;\gamma}) \tag{17}$$

and

$$R_{M\alpha\beta}=(P^{\gamma}{}_{\delta\beta}P^{\delta}{}_{\alpha\gamma}-P^{\gamma}{}_{\delta\gamma}P^{\delta}{}_{\alpha\beta}) \tag{18}$$

as the gravitational and material parts of the Ricci tensor. Clearly,

$$R_{\alpha\beta}=R_{G\alpha\beta}+R_{M\alpha\beta} \tag{19}$$

The following expressions of the Ricci tensor and its distinct parts in terms of the fundamental tetrad are easily verified:

$$R_{\alpha\beta}=\phi^{\rho}_{a}(\phi^{a}_{\alpha;\rho\beta}-\phi^{a}_{\alpha;\beta\rho}) \tag{20}$$

$$R_{G\alpha\beta}=\phi^{\rho}_{a;\beta}\phi^{a}_{\alpha;\rho}-\phi^{\rho}_{a;\rho}\phi^{a}_{\alpha;\beta}+\phi^{\rho}_{a}\phi^{a}_{\alpha;\rho\beta}-\phi^{\rho}_{a}\phi^{a}_{\alpha;\beta\rho} \tag{21}$$

$$R_{M\alpha\beta}=\phi^{\rho}_{a;\rho}\phi^{a}_{\alpha;\beta}-\phi^{\rho}_{a;\beta}\phi^{a}_{\alpha;\rho} \tag{22}$$

The partition of the Ricci tensor induces a decomposition of the Einstein tensor:

$$G_{\alpha\beta}=G_{G\alpha\beta}+G_{M\alpha\beta} \tag{23}$$

with

$$G_{G\alpha\beta}=R_{G(\alpha\beta)}-\tfrac{1}{2}g_{\alpha\beta}R^{\rho}_{G\rho} \tag{24}$$

and

$$G_{M\alpha\beta}=R_{M(\alpha\beta)}-\tfrac{1}{2}g_{\alpha\beta}R^{\rho}_{M\rho} \tag{25}$$

where brackets denote symmetrization over the two indices.

We are now in the position to write down explicitly the field equations (8) and (9) in terms of the components of the fundamental tetrad:

$$\phi^{\rho}_{a;(\beta}\phi^{a}_{\alpha);\rho}-\phi^{\rho}_{a;\rho}\phi^{a}_{(\alpha;\beta)}+\phi_{a\rho}\phi^{a\rho}_{(\alpha;\beta)}-\phi^{\rho}_{a}\phi^{a}_{(\alpha;\beta)\rho}$$
$$-\tfrac{1}{2}\,g_{\alpha\beta}(\phi^{\rho}_{a;\gamma}\phi^{a\gamma}{}_{;\rho}-\phi^{\rho}_{a;\rho}\phi^{a\gamma}{}_{;\gamma}+\phi^{\rho}_{a}\phi^{a\gamma}{}_{;\rho\gamma}-\phi^{a\gamma}{}_{;\gamma\rho})=\kappa T_{G\alpha\beta} \tag{26a}$$

$$\phi^{\rho}_{a;\rho}\phi^{a}_{(\alpha;\beta)}-\phi^{\rho}_{a;(\beta}\phi^{a}_{\alpha);\rho}-\tfrac{1}{2}g_{\alpha\beta}(\phi^{\rho}_{a;\rho}\phi^{a\gamma}{}_{;\gamma}-\phi^{\rho}_{a;\gamma}\phi^{a\gamma}{}_{;\rho})=\kappa T_{M\alpha\beta} \tag{26b}$$

and

$$\phi^{\rho}_{a}(\phi^{a}_{\alpha;\rho\beta}-\phi^{a}_{\alpha;\beta\rho})-\tfrac{1}{2}g_{\alpha\beta}\phi^{\rho}_{a}(\phi^{a\gamma}{}_{;\rho\gamma}-\phi^{a\gamma}{}_{;\gamma\rho})=\kappa T_{T\alpha\beta} \tag{27}$$

where equation (26a) is the tensorial expression of the gravitational energy, as a function of metric elements, that we are looking for. The fundamental tetrad, in turn, determines the metric tensor by equation (10).

Material and gravitational energy-momentum are now independent physical quantities, related by the four constraints of the conservation law

$$(T_M^{\alpha\beta} + T_G^{\alpha\beta})_{;\beta} = 0$$

which reflect the Bianchi identities.

5. LORENTZ INVARIANCE

Having established the field equations (26) and a tensorial expression for the gravitational energy (26a), which are clearly covariant under the group of general coordinate transformations, we now turn to the question of their behavior under tetrad transformations.

The most general transformation which preserves the orthonormality relation (10) is a Lorentz transformation. First, consider the case of a global Lorentz transformation

$$\phi'^{\alpha}_{a} = L_a{}^b \phi_b{}^\alpha \tag{28}$$

where the matrix L is constant. Clearly, the tensorial components of the Ricci coefficients of rotation P remain invariant under such a transformation, and hence the gravitational and material parts (24) and (25) of the Einstein tensor remain invariant as well. The global orientation of the tetrad throughout spacetime is therefore inconsequential so far as the distribution of the total energy-momentum and its partition between gravitational and matter energy-momentum is concerned.

The situation is different when a local (coordinate-dependent) Lorentz transformation is applied to the tetrad. While equation (20) is still invariant, equations (21) and (22) are not. That is, the local relative orientation of the tetrads determine the partition of the total energy-momentum between the gravitational and matter parts.

Thus, for a given background metric g and the total energy-momentum tensor $T_T{}^{\alpha\beta}$ associated with it, the 16 components of the tetrad are subject to the ten algebraic constraints

$$\phi^{\alpha}_{a} \phi^{a\beta} = g^{\alpha\beta} \tag{29}$$

which leaves six degrees of freedom in the choice of the tetrad. This grade of freedom corresponds to the six parameters—functions of the coordinates—that define a local Lorentz transformation

$$\phi'^{\alpha}_{a} = L_a{}^b(x) \phi_b{}^\alpha \tag{30}$$

The latter leaves the Einstein tensor, and hence the total energy-momentum, invariant, but changes their partition into material and gravitational parts.

6. GRAVITATIONAL ENERGY DUE TO A SPHERICALLY SYMMETRIC SOURCE

We now present an exact solution of the field equations corresponding to a spherically symmetric source. Let the source be of mass M and radius R, and assume that the total energy-momentum (material plus gravitational) vanishes in the exterior of the source, viz.,

$$T_T^{\alpha\beta} = T_M^{\alpha\beta} + T_G^{\alpha\beta} = 0 \qquad (r > R) \tag{31}$$

Under these circumstances, the unique spherically symmetric solution for the metric of the underlying manifold is the Schwarzschild metric

$$g_{\alpha\beta} = \mathrm{diag}(\Lambda, -\Lambda^{-1}, -r^2, -r^2 \sin^2\theta) \tag{32}$$

with

$$\Lambda = 1 - 2MG/r \tag{33}$$

The general solution for the fundamental tetrad field is ϕ' as given by (30), with ϕ being a particular tetrad compatible, through equation (29), with the Schwarzschild metric (32).

The choice of the solution suitable for the present case is guided by imposing two physical requirements:

1. The gravitational energy-momentum tensor must be diagonal in the Schwarzschild coordinates.
2. The fundamental tetrad field must be asymptotically parallel at spatial infinity.

The first requirement reflects spherical symmetry, and is expressed in a system of six differential equations for the six parameters of the local Lorentz transformation in equation (30), while the second, which constitutes a set of boundary conditions, is the embodiment of the assumption that the deviation from parallelism is traced to the curvature of spacetime.

A solution satisfying these postulates is found to be

$$\begin{aligned}
\phi_0^\alpha &= (\Lambda^{-1/2}, 0, 0, 0)\\
\phi_1^\alpha &= (0, \Lambda^{1/2}\sin\theta\cos\phi, r^{-1}\cos\theta\cos\phi, -r^{-1}\sin^{-1}\theta\sin\phi)\\
\phi_2^\alpha &= (0, \Lambda^{1/2}\sin\theta\sin\phi, r^{-1}\cos\theta\sin\phi, -r^{-1}\sin^{-1}\theta\cos\phi)\\
\phi_3^\alpha &= (0, \Lambda^{1/2}\cos\theta, -r^{-1}\sin\theta, 0)
\end{aligned} \tag{34}$$

To visualize the physical meaning of this tetrad, consider a fleet of spaceships, each carrying an orthonormal triad, dispersing radially from the star, and remaining at rest after rotating themselves through the angles required in order for them to stay "parallel" to each other. In flat space they will constitute the spatial part of a parallel tetrad, with $\phi_a{}^\alpha{}_{;\beta}=0$ and $T_G{}^{\alpha\beta}=0$. In the presence of the massive star, however, they will deviate from parallelism by the curvature of spacetime, while $T_G{}^{\alpha\beta}$ is assumed to attain the value of the gravitational energy-momentum associated with the curvature. Clearly, any tetrad obtained by applying a global Lorentz transformation to this tetrad may serve as well.

The four nonvanishing components of the gravitational energy-momentum tensor obtained replacing (34) in (26a) are

$$T^0_{G0}=-(1/\kappa)(1-\Lambda^{1/2})^2r^{-2} \tag{35a}$$

$$T^1_{G1}=-(1/\kappa)[(1-\Lambda^{1/2})^2r^{-2}+(1-\Lambda^{-1/2})2MG/r^3] \tag{35b}$$

$$T^2_{G2}=T^3_{G3}=-(1/\kappa)(1-\Lambda^{-1/2})MG/r^3 \tag{35c}$$

It represents a spherically symmetric distribution of the gravitational energy approaching zero as r tends to infinity, or when M tends to zero.

Evaluating the invariant spatial integral of the gravitational energy expressed by (35a) from the surface of the source ($r=R$) to infinity, one obtains

$$E=\int_R^\infty \sqrt{-g}\,\Phi_0^\alpha T^0_{G\alpha}\,d^3x=-M(1-\Omega)/(1+\Omega) \tag{36}$$

where $\Phi_0{}^\alpha$ is a timelike unit vector normal to the spatial hypersurface in which the source is at rest, and with

$$\Omega=(1-2GM/R)^{1/2}$$

Expanding the above expression in powers of $2GM/R$ and discarding terms in the third and higher powers, we find

$$E\approx-(1/2)GM^2/R$$

which is precisely the Newtonian value for the gravitational energy of a spherical shell of radius R and mass M. (Notice that for such a configuration of mass, the total amount of Newtonian gravitational energy may be ascribed to the external field, the latter being independent of the radial distribution of matter within the source.)

In the special case of the sun, taking

$$M=2\times10^{33}\text{ g}$$

as the sun mass,

$$G=7.425\times10^{-29}\text{ c g}^{-1}$$

as the Newton gravitational constant, and

$$R=7\times10^{10}\text{ c}$$

as the sun's radius, the relativistic value obtained from equation (36) is

$$E = -2.121432\cdots\times 10^{27}\text{ g}$$

which agrees with the Newtonian value

$$E_{\mathrm{N}} = -2.121428\cdots\times 10^{27}\text{ g}$$

up to the sixth digit.

7. COMMENTS AND REMARKS

The covariant conservation law of energy-momentum, expressed by the covariant divergencelessness of the energy-momentum tensor, results in a global continuity equation valid in a preferred subclass of the geodesic coordinates—the geodesic-nonrotating frames. This global continuity equation leads, in turn, to a physically meaningful integral conservation law. Thereby the need for a kind of energy external to the energy-momentum tensor, intended to complement the latter to a globally divergenceless quantity, is removed. We are therefore motivated to look for a tensorial expression of the gravitational energy as an integral part of the (now construed as total) energy-momentum tensor that appears in the Einstein field equation.

In this context a tensorial expression for the gravitational energy-momentum as a function of a metric tetrad is here proposed. It arises in a natural way, following the replacement of the metric tensor by a metric tetrad as the fundamental element of the gravitational field. The ensuing energy of the external gravitational field of a star calculated from this tensorial expression results in high agreement with the corresponding Newtonian value. It is worth noticing, that this agreement arises spontaneously without the introduction of an ad hoc constant or other coercive means.

The Einstein field equations are preserved in their original form, with a gravitational contribution to the sources now assumed. They are cast as a quasilinear second-order system of 16 independent differential equations for the same number of components of the tetrad, when the latter replaces the metric, with the gravitational and material energy-momentum tensors regarded as independent data.

In this approach, the gravitational energy is no longer a nonlocalized evasive quantity of perplexing physical meaning. On the contrary, its tensorial, and consequently localized, nature present it as a totally regular form of energy.

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